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Existence & Uniqueness

Consider $\dot{x} = x^{1/3}$; $x(0)=0$ $\quad (x^{1/3} \longleftrightarrow \sqrt[3]{x})$

analytically solvable : $\dfrac{dx}{dt} = x^{1/3} \xrightarrow[\text{of variables}]{\text{separation}} \displaystyle\int \frac{dx}{x^{1/3}} = \int dt + c$

$$\Rightarrow \int x^{-1/3} dx = t + c \quad \Rightarrow \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} = t + c$$

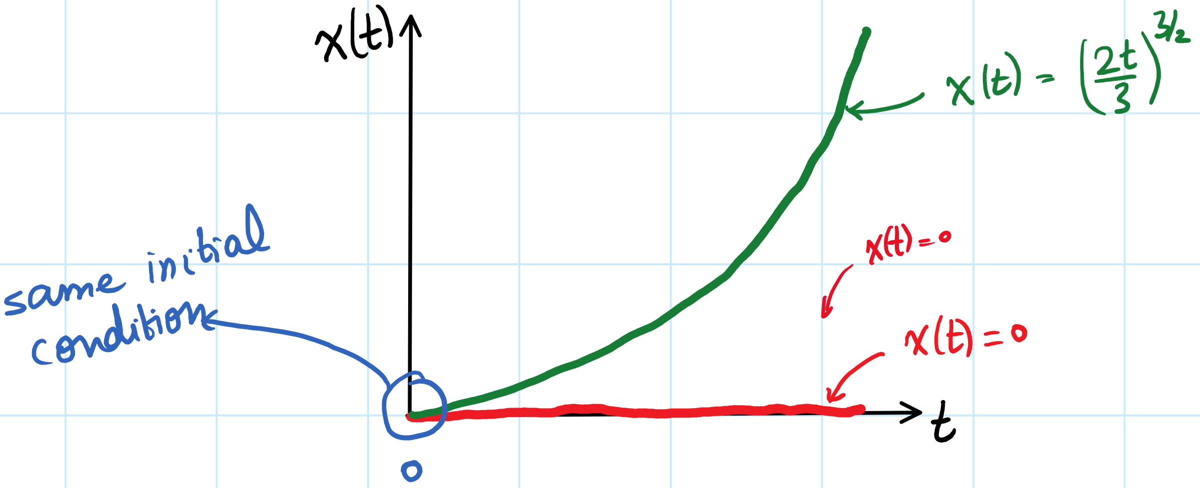
$$\Rightarrow \frac{x^{2/3}}{2/3} = t + c \quad \Rightarrow x^{2/3} = \frac{2}{3}(t+c)$$

$$\Rightarrow x(t) = \left[\frac{2}{3}(t+c)\right]^{3/2}$$

invoke initial condition : $x(0)=0 \rightarrow 0 = c^{3/2} \Rightarrow c=0$

$\therefore$ $\boxed{x(t) = \left(\frac{2}{3}t\right)^{3/2}}$ is $\underline{\underline{a}}$ solution.

But : $x(t) = 0$ for all time is $\underline{\text{also}}$ a solution!



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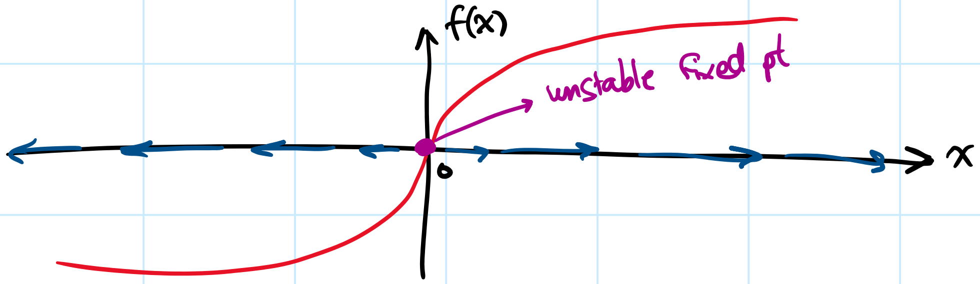
Phase Portrait of $\dot{x} = f(x)$; $f(x) = x^{1/3}$

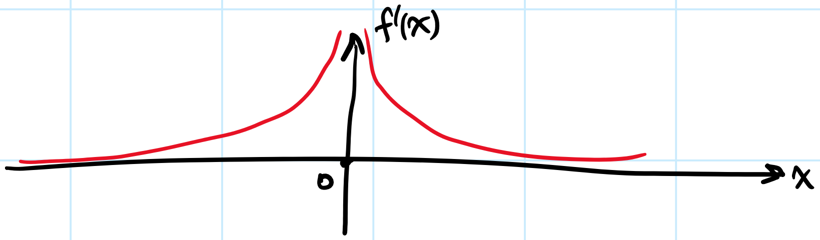


Linearized stability analysis @ the fixed pt $x=0$

$f(x) = x^{1/3} \Rightarrow f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3x^{2/3}}$

$f'(0) \longrightarrow \infty$

"very" unstable



Existence & Uniqueness Thm: Consider $\dot{x} = f(x)$; $x(0) = x_0$

If f and f' are continuous functions

$\Rightarrow \exists!$ solution ($\exists!$: There exist 1 solution)

Fact: Even if the solution exists, it does not mean that it exists for all time!

example: Consider $\dot{x} = f(x)$; $x(0) = 0$ $f(x) := 1 + x^2$

analytical solution: $\frac{dx}{dt} = 1 + x^2 \longrightarrow \int \frac{dx}{1+x^2} = \int dt + c$

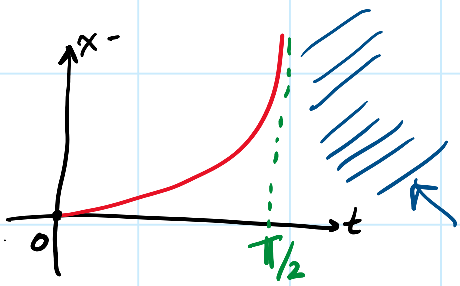
$\Rightarrow \tan^{-1}(x) = t + c$

$\Rightarrow x(t) = \tan(t + c)$

Invoke initial condition: $x(0) = 0 \Rightarrow 0 = \tan(0 + c)$

$\Rightarrow \tan c = 0 \Rightarrow c = 0$

$\therefore \boxed{x(t) = \tan t}$ (I)



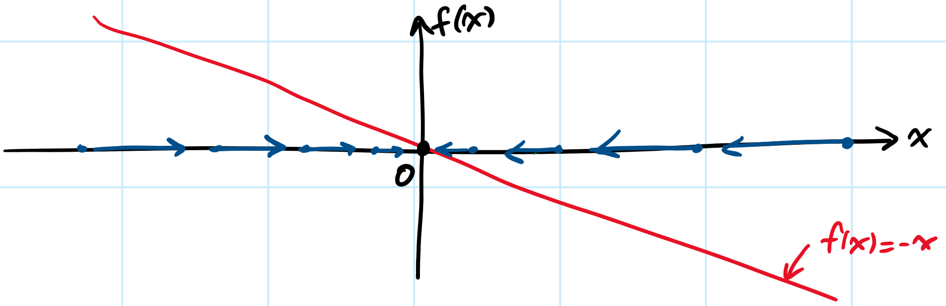
In finite time $(t = \pi/2)$, the dynamics "blow up"!

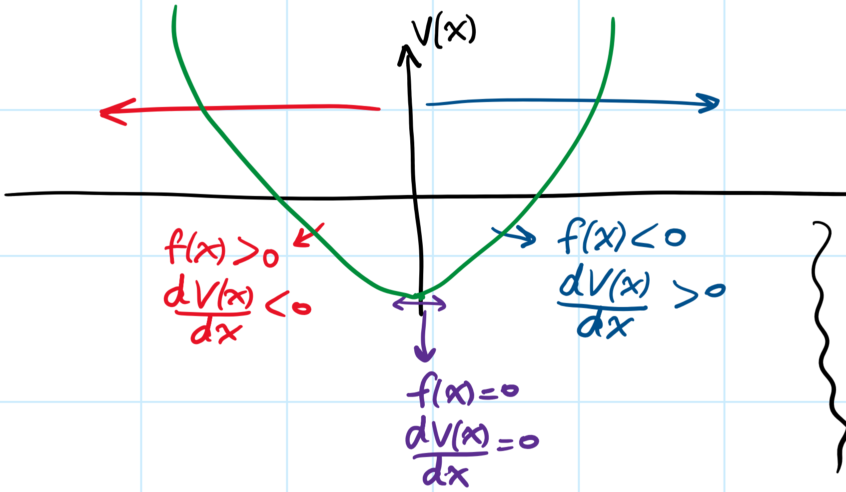
solution doesn't exist! $(t \geq \pi/2)$

Potentials

Make the "bucket" analogy mathetically precise.

Consider $\dot{x} = -x$; $x(0) = x_0$ $\quad (f(x) = -x)$





$$\frac{dV(x)}{dx} = -f(x)$$

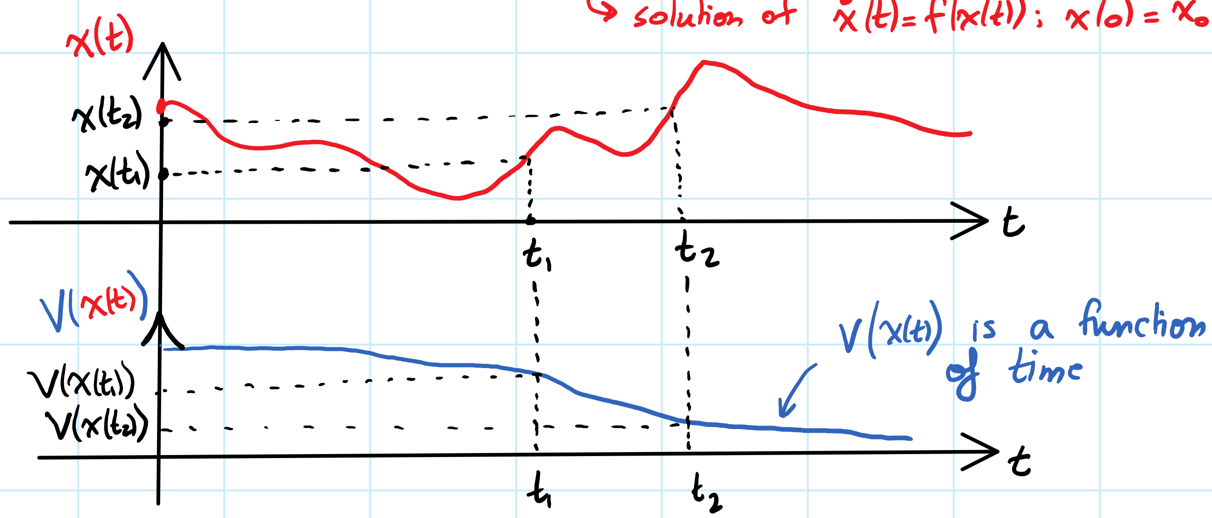
Let $x(t)$ be a trajectory (solution) for $\dot{x}(t) = f(x(t))$; $x(0) = x_0$

Aside: What does "evaluate V along a trajectory?

V is a function of x: it takes a number to spit another number $V(x)$

But V can be evaluated along $x(t)$ for all time $\rightarrow V(x(t))$

$\hookrightarrow$ solution of $\dot{x}(t) = f(x(t))$; $x(0) = x_0$.



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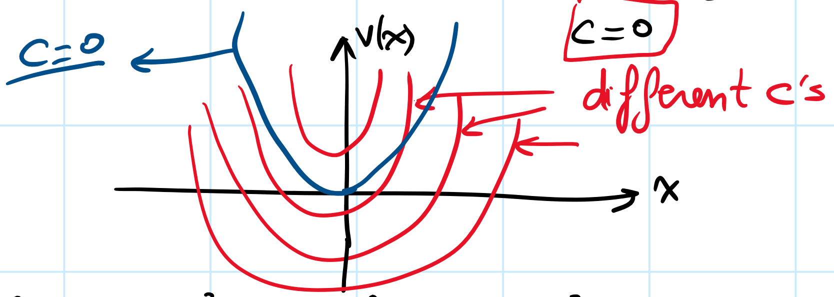
Fact: The value of V evaluated along the trajectory decreases in time (i.e. ball goes down)

Because: $\frac{dV(x(t))}{dt} = \underbrace{\frac{dV(x)}{dx}}_{-f(x)} \cdot \underbrace{\frac{dx}{dt}}_{f(x)} = -f^2(x) \leq 0$

Examples of finding potentials

Example 1: $\dot{x} = -x$ $\frac{dV(x)}{dx} = -f(x) \Rightarrow V(x) = -\int f(x)dx + c$

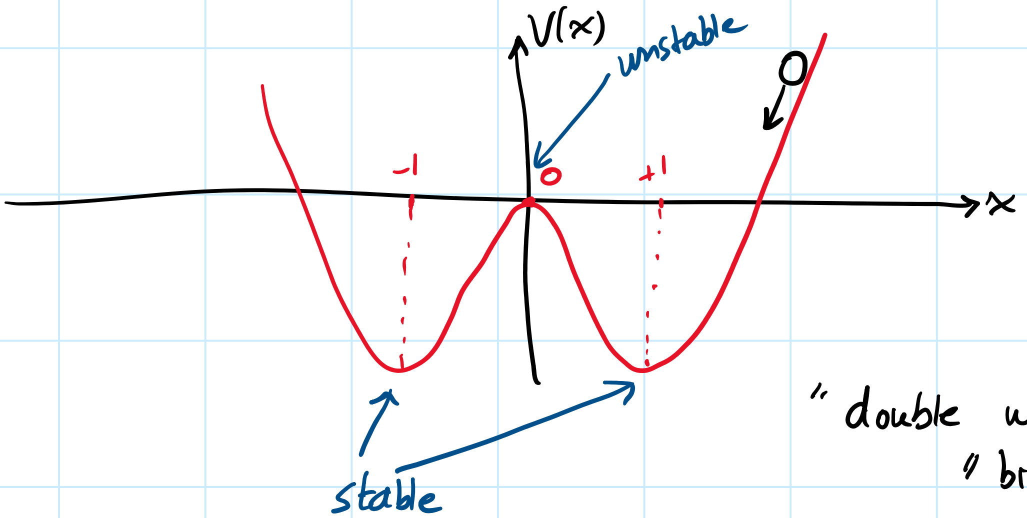
$$V(x) = \int -x\,dx + c = -\frac{x^2}{2} + c$$



Example 2: $\dot{x} = x - x^3$ $f(x) = x - x^3$

$\boxed{\frac{dV(x)}{dx} = -f(x)} \Rightarrow V(x) = -\int f(x)dx + \cancel{c}^{\,0}$

$$= -\int (x - x^3)dx = -\frac{x^2}{2} + \frac{x^4}{4}$$



"double well potential"
"bistable"